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LETTER TO THE EDITOR

The interaction $V(r)=-Ze^2/(r+\beta)$ and the confluent Heun equation

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Abstract. The Schrödinger equation with interaction $-Ze^2/(r+\beta)$ is shown, for general values of the parameters, to be reducible to the confluent Heun equation.

For the potential $-Ze^2/(r+\beta)$, the radial Schrödinger equation is

$$y''+[2E+(2Ze^2/(r+\beta))-(l(l+1)/r^2)]y(r)=0. \tag{1}$$

This interaction has recently attracted some attention in connection with, among other things, the model of the potential due to a smeared charge. See de Meyer and van den Berghe (1990). If $l=0$ (or the physically inadmissible value $l=-1$), this equation can be solved in terms of confluent hypergeometric functions. Otherwise, the problem is reduced to solutions of the confluent Heun equation.

This last mentioned equation has three singularities in all. Two of these are regular and the remaining singularity is irregular of the second type. The confluent Heun equation may be characterized by the Ince symbol $[0, 2, 1_2]$, Ince (1926). See also the references in Marcilhacy and Pons (1985) for example. In its canonical form, this equation may be written

$$z(1-z)w''+[c-(a+b+1)z-kh^2z^2]w'-(ab+kz)w=0 \tag{2}$$

and its normal form is

$$\begin{aligned} z^2(1-z)^2y''=\{&(k^2h^2z^4/4)+k[(h^2(a+b+1)/2)-1]z^3\\ &+[(a(a+1)+b(b+1))/2-(kh^2(1+c/2)/2)]z^2\\ &+[ab-(c(a+b+1)/2)]z+c[(c/2-1)/2]\}y \end{aligned} \tag{3}$$

where

$$y=\exp(kh^2z/2)z^{c/2}(z-1)^{(a+b+1-c+kh^2)/2}w. \tag{4}$$

In (1), put $r=-\beta z$ and obtain

$$\begin{aligned} z^2(1-z)^2y''=\{&2\beta^2Ez^4-2\beta(Ze^2+2\beta E)z^3+[2\beta(Ze^2+2\beta E)-l(l+1)]z^2\\ &+2l(l+1)z-l(l+1)\}y \end{aligned} \tag{5}$$

after a little rearrangement. It is then immediately evident that (5) is of the same form as (3) and that (1) can be represented as a confluent Heun equation.

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